Lecture 10

Tuesday, September 27, 2016 8:54 AM

3.4 THE CHAIN RULE

Q $F(x) = \sqrt{x^2+1}$, $F'(x) = ?$

Observe F is a composite func.

$y = f(u) = \sqrt{u} = u^{1/2}$ →

$u = g(x) = x^2+1$ →

Then $F(x) = f(g(x))$ i.e $F = f \circ g$

CHAIN RULE

$F(x) = f(g(x))$

$f, g \equiv$ diff functions, $F = f \circ g$

Then F is a diff function and

F' is given by the following fmla:

$F'(x) = f'(g(x)) \cdot g'(x)$

product

If $y = f(u)$, $u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Ex $F(x) = \sqrt{x^2+1}$

$F(x) = f(g(x))$, $f(u) = \sqrt{u} = u^{1/2}$

$g(x) = x^2+1$

$f'(u) = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}}$

$g'(x) = 2x$

$F'(x) = f'(g(x)) \cdot g'(x)$

$= \frac{1}{\cancel{2}\sqrt{x^2+1}} \cdot \cancel{2}x = \frac{x}{\sqrt{x^2+1}}$

$$\frac{d}{dx} \underbrace{f}_{\text{outer}} \underbrace{(g(x))}_{\text{inner}} = \underbrace{f'}_{\text{deriv of out fn}} \underbrace{(g(x))}_{\text{evaluated at the inner fn}} \overbrace{\cdot}^{\text{product}} \underbrace{g'(x)}_{\text{deriv of inner fn}}$$

Ex 1) $y = \sin(x^2)$

2) $f(x) = \sin^2 x$

1) Outer is the sine fnc.

Inner fnc is the squaring fnc.

$$\frac{dy}{dx} = \frac{d}{dx} \overbrace{\sin}^{\text{out}} \overbrace{(x^2)}^{\text{inn}}$$

eval at inner

$$= \underbrace{\cos(x^2)}_{\text{der of out}} \cdot \underbrace{2x}_{\text{der of inner}} = 2x\cos(x^2)$$

2) $f(x) = \sin^2 x = (\sin x)^2$

$$f'(x) = \underbrace{2 \cdot \sin x}_{\text{derivative of outer functi evaluated at inner fn}} \cdot \cos x = 2\sin x \cdot \cos x$$

$$= \sin(2x)$$

Special Case of Chain Rule

$y = [g(x)]^n$, $y = f(u) = u^n$, where $u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = nu^{n-1} \cdot \frac{du}{dx}$$

$$= \underbrace{n[g(x)]^{n-1}} \cdot \underbrace{g'(x)}$$

Power rule combined w/ Chain rule.

Ex Find $f'(x)$ if $f(x) = \dfrac{1}{\sqrt[3]{x^2+x+1}}$

$$f(x) = \frac{1}{(x^2+x+1)^{1/3}} = (x^2+x+1)^{-1/3}$$

$$f'(x) = -\frac{1}{3}(x^2+x+1)^{-1/3-1} \cdot \frac{d}{dx}(x^2+x+1)$$

$$= -\frac{1}{3}(x^2+x+1)^{-4/3} \cdot (2x+1)$$

Ex $g(t) = \left(\dfrac{t-2}{2t+9}\right)^9$

$$g'(t) = 9\left(\frac{t-2}{2t+9}\right)^8 \cdot \frac{d}{dt}\left[\frac{t-2}{2t+9}\right]$$

$$= 9\left(\frac{t-2}{2t+9}\right)^8 \cdot \left[\frac{\overbrace{(2t+9)\cdot 1}^{2t+9} - \overbrace{(t-2)\cdot(2)}^{-(2t-4)}}{(2t+9)^2}\right] \quad = 2t+9-2t+4 = 13$$

$$= 9\left(\frac{t-2}{2t+9}\right)^{8} \cdot \frac{13}{(2t+9)^{2}} \qquad \frac{(t-2)^8}{(2t+9)^8}$$

$$= 117\,\frac{(t-2)^8}{(2t+9)^{10}}$$

Ex $f(x) = \sin(\cos(\tan x))$

$$f'(x) = \underbrace{\cos}_{\substack{\text{deriv}\\ \text{of out}}}\underbrace{(\cos(\tan x))}_{\substack{\text{evaluated}\\ \text{at inner}}} \cdot \underbrace{\frac{d}{dx}\left[\cos(\tan x)\right]}_{\text{deriv of inner}}$$

$$= \cos(\cos(\tan x)) \cdot -\sin(\tan x) \cdot \sec^2 x$$

Ex $y = \sqrt{\sec(x^3)}$, $\frac{dy}{dx} = ?$

$$= \left[\sec(x^3)\right]^{1/2}$$

$\frac{d}{dx}[\sec x] = \sec x . \tan x$

$$\frac{dy}{dx} = \frac{1}{2}\left[\sec(x^3)\right]^{-1/2} \cdot \frac{d}{dx}\left[\overset{\text{out}}{\widehat{\sec}}(\underbrace{x^3}_{\text{inner}})\right]$$

$$= \frac{1}{2}\left[\sec(x^3)\right]^{-1/2} \cdot \sec(x^3) . \tan(x^3) . 3x^2$$

Simplify DIY

3.5 Implicit Differentiation.

So far, we have found derivatives of functions of the form $y = f(x)$

$y = \sin(x^2)$, $y = \sqrt{x^2+1}$

Some funcs however are defined implicity by a relation between x and y

$x^3 + y^3 = 6xy$ (Folium of Descartes)

The method of implicit differentiation.

In this discussion, y is a function of x.

Ex If $x^2 + y^2 = 25$, find $\frac{dy}{dx}$.

1) Differentiate both sides of the equation with respect to x.

$$\frac{d}{dx}[x^2+y^2] = \frac{d}{dx}[25]$$

$$\Rightarrow \frac{d}{dx}[x^2] + \frac{d}{dx}[y^2] = 0$$

$$\Rightarrow 2x + \frac{d}{dx}[y^2] = 0.$$

Since y is a function of x, to find $\frac{d}{dx}[y^2]$ we use Chain Rule,

$$\frac{d}{dx}[y^2] = 2y^1 \cdot \frac{dy}{dx}$$

$$\frac{d}{dx}[xy]$$

$$= \frac{d}{dx}(x)\, y + x\, \frac{dy}{dx}$$

$\frac{d}{dx}$ [...]

$$\frac{d}{dx}[y^2] = \underbrace{2y^1}_{} \cdot \frac{dy}{dx}$$

power rule combined w/ chain rule

$$\frac{d}{dx}[y^2] = \frac{d}{dy}[y^2] \cdot \frac{dy}{dx} = 2y\frac{dy}{dx}$$

$$= \frac{d}{dx}(x)\cdot y + x \cdot \frac{dy}{dx}$$

$$= 1 \cdot y + x\frac{dy}{dx}$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2y\frac{dy}{dx} = -2x$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

Ex Find the tangent line to the curve $x^3 + y^3 = 6xy$ at pt $(3,3)$.

$$\frac{d}{dx}[x^3+y^3] = \frac{d}{dx}[6xy]$$

$$\Rightarrow 3x^2 + \frac{d}{dx}[y^3] = 6\frac{d}{dx}[xy]$$

$$\Rightarrow 3x^2 + 3y^2 \cdot \frac{dy}{dx} = 6\left[1 \cdot y + x \cdot \frac{d}{dx}[y]\right]$$

$$\Rightarrow 3x^2 + 3y^2 \cdot \frac{dy}{dx} = 6\left[y + x\frac{dy}{dx}\right]$$

$$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

$$\Rightarrow 3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$$

$$\Rightarrow \frac{dy}{dx}\left[3y^2 - 6x\right] = 6y - 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x} \quad ✓$$

$$m = \frac{dy}{dx}\Big| \quad = -1$$

$$f(x) = f(xf(x^2))$$

$$\underbrace{f'(x)}_{2} = f'(\underbrace{xf(x^2)}_{2}) \cdot \left[f(x^2) + x \cdot f'(\underbrace{x^2}_{2}) \cdot 2x\right]$$

$x = 2$

$$f'(2) = f'(2 \cdot f(4)) \cdot \left[f(4) + 2 \cdot f'(4) \cdot 4\right]$$

$$f'(2) = f'(2 \cdot 3) \cdot \left[3 + 2 \cdot 3 \cdot 4\right]$$

$m = \left.\frac{dy}{dx}\right|_{(3,3)} = -1$

$f'(2) = f'(2.3) \cdot \left[3 + 2 \cdot 3 \cdot 4\right]$

$= f'(6) \cdot [27]$

$f'(2) = 27$

$\sin(\cos(\tan x))$

$\frac{d}{dx}\left[\cos(\tan x)\right] = -\sin(\tan x) \cdot \sec^2 x$